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Variational method for the nonlinear dynamics of an elliptic magnetic stagnation line

A.H. Khater[1,2,a], D.K. Callebaut[2], M.A. Helal[3], and A.R. Seadawy[1]

[1] Mathematics Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
[2] Department Natuurkunde, Campus Groeneborger U 306, University of Antwerp (UA), 2020 Antwerpen, Belgium
[3] Mathematics Department, Faculty of Science, Cairo University, Giza, Egypt

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Abstract. The nonlinear evolution of the kink instability of a plasma with an elliptic magnetic stagnation line is studied by means of an amplitude expansion of the ideal magnetohydrodynamic equations. Wahlberg et al. [12] have shown that, near marginal stability, the nonlinear evolution of the stability can be described in terms of a two-dimensional potential $U(X,Y)$, where X and Y represent the amplitudes of the perturbations with positive and negative helical polarization. The potential $U(X,Y)$ is found to be nonlinearly stabilizing for all values of the polarization. In our paper a Lagrangian and an invariant variational principle for two coupled nonlinear ordinal differential equations describing the nonlinear evolution of the stagnation line instability with arbitrary polarization are given. Using a trial function in a rectangular box we find the functional integral. The general case for the two box potential can be obtained on the basis of a different ansatz where we approximate the Jost function by polynomials of order n instead of a piecewise linear function. An example for the second order is given to illustrate the general case. Some considerations concerning solar filaments and filament bands (circular or straight) are indicated as possible applications besides laboratory experiments with cusp geometry corresponding to quadripolar cusp geometries for some clouds and thunderstorms.

PACS. 02.30.Xx Calculus of variations – 52.25.Xz Magnetized plasmas – 52.55.-s Magnetic confinement and equilibrium

1 Introduction

In order to understand the anomalous plasma stability observed in various fusion devices based on the Z-pinch, such as for instance the Extrap configuration, the dense Z-pinch, and the fiber pinch, it is necessary to extend the theory of the internal $m = 1$ (kink) instability beyond the stabilizing assumptions of the linear, ideal magnetohydrodynamic (MHD) model [1–6]. It is well-known that such plasma is unstable within ideal MHD theory [4,5,7,8]. Furthermore, two important properties of the eigenmodes of this instability are that (i) they are localized arbitrarily close to the stagnation line (the stagnation line in a magnetized plasma is a line along which the magnetic field vanishes) in the limit of short axial wavelengths, and (ii) they involve a displacement, or kinking, of the stagnation line [7,8]. The instability can therefore, in a sense, be regarded as a property of the stagnation line itself.

In general, the nonlinear evolution (NLE) of the kink instability of plasma with an elliptic magnetic stagnation line (EMSL) is studied by means of an amplitude expansion of an ideal MHD equations. Cylindrically symmetric plasma with circular field lines is used to model the magnetic field geometry close to the stagnation line. Due to the symmetry with respect to $\pm z$, the linear stability problem of such a system has a two-folded degeneracy, with equal eigenvalues for helical kink perturbations with positive and negative polarization. Wahlberg et al. in a series of papers [8–12] have shown that, near marginal stability, the NLE of the instability can be described in terms of a two-dimensional potential $U(X,Y)$, where X and Y represent the amplitudes of the perturbations with positive and negative helical polarization. They obtained the NLE of the stagnation line instability with either positive or negative helical polarization, they constructed the energy $K + U = constant$, where K is the kinetic energy, and U the potential energy for the specific physics at hand and they discussed the stability from the energy integral.

We started on one hand from the evolution equation (EE) obtained in [12] and on the other hand from the existence of a Lagrangian and an invariant variational principle (IVP) (i.e. in the sense of the inverse problem of calculus of variations (CV) through deriving the functional integral (FI) corresponding to given coupled

[a] e-mail: khater_ah@yahoo.com

nonlinear ordinal differential equations (CNLODEs). The obtained results turned out to be in agreement with those ones given in [13–21]. Moreover, the first and the second variations of the obtained FI (i.e. direct problem of the CV) are also carried out and led again to be an extension of the results of the stability criteria given in Wahlberg et al. [8–12] to include the bistable systems (including nonlinearity and solitons) [12]. The invariance identities involving the Lagrangian L and the generators of the infinitesimal Lie group of transformations will be utilized for writing down their first integrals via Noether's theorem (Logan [23,24]). We formulate a variational principle (VP) for the CNLODEs describing the NLE of stagnation line instability. We shall demonstrate the simplest example of the application of this technique, taking the box-shaped initial pulse and a Jost functions.

This paper is organized as follows: in Section 2, a brief outline is given of the linear stability properties of the EMSL model in incompressible, ideal MHD. Some general properties of the equations describing the nonlinear dynamics of the stagnation line are discussed. The existence and formulation of the variational principle (VP) for the resulting nonlinear evolution equation (NLEE). The invariance identities [22] involving the Lagrangian L and the generators of the infinitesimal Lie group of transformations have been used to obtain the first integrals via Noether's theorem [13–21,23,24]. Further, through the repeated application of invariance under the transformation, the exact solution of the EE, which has been generated for various choices of the parameters involved, is given. In Section 3, we formulate the VP for the two CNLODEs describing the NLE of stagnation line instability and we construct the Lagrangian L [25–28]. In Section 4, we demonstrate the simplest example of the application of this technique, taking the box-shaped initial pulse and an ansatz based on a linear Jost function in the region of localization of the box. In Section 5, we consider in more details the limiting case of the above mentioned box with the phase jump equal to π, i.e., a combination of two boxes of opposite signs, the total area of the initial pulse being thus zero. We develop a variational approximation for finding the eigenvalues of this pulse, the Jost function being approximated by a piecewise linear ansatz, which has two variational parameters. In Section 6, we approximate the Jost functions by polynomials of order n instead of piecewise linear functions. Section 7 indicates the possibility of an application to some solar filaments. Finally the paper ends with a conclusion in Section 8.

2 Solution of linear theory

We first discuss the general structure of the NLE of the linearly unstable internal $m = 1$ kink modes in a pure Z-pinch. As shown in a number of previous studies of nonlinear, ideal MHD phenomena, the nonlinear dynamics of a nearly marginal, nonresonant mode is in general described by an equation of the type [8–12,28,29]

$$\frac{d^2X}{d\tau^2} + D_1X + D_3X^3 = 0, \tag{1}$$

where $X(t)$ stands for some suitable quantity representing the amplitude of the mode $X(t)\cos(\varphi + kz)$. Thus $X(t)$ denotes the amplitude of the plasma helix (normalized to the plasma radius). $k \approx k_c$ with k_c the critical wave number at the marginal point.

The linear dispersion relation ($\omega^2 = D_1$ for the mode $m = 1$) is obtained for $D_3 = 0$. To see the influence of the nonlinear term D_3X^3 it is important to have a clear idea of the ordering of the quantities. Using the symbol δ to denote the order of the mode amplitude X, the time scale τ in this equation is proportional to $\tau = \delta t$. Furthermore, D_1, the linear dispersion function of the nearly marginal $m = 1$ kink mode, is of order δ^2, which in terms of the wave number means that $k - k_c = O(\delta^2)$. $D_3 = D_3(k_c)$ is a number of order unity which determines the nonlinear properties of the mode. If D_3 is positive, the nonlinear term stabilizes an unstable ($D_1 < 0$) mode at a finite amplitude, whereas a negative value of D_3 implies that the mode grows explosively in the nonlinear regime. The quantities $D_1(k_c)$ and $D_3(k_c)$ may be calculated within incompressible ideal MHD [12], using the helical equilibrium model developed in reference [9]. For illustrations we shall use $D_1 = -1$ and $D_3 = 1.5$.

We discuss the IVP for equation (1) and we find the exact solution. Now the existence of the VP is proved as follows: consider

$$N(\tau, X, X', X'') = \frac{d^2X}{d\tau^2} + D_1X + D_3X^3 = 0. \tag{2}$$

The shape of this equation is clear: without the term in X^3, it corresponds to the linearized dispersion relation ($d^2/d\tau^2 \to \omega^2$). Extending it to the nonlinear analysis we have no term in X^2 as we are near an extremum. The next nonlinear term is in X^3. For any nonlinear operator $N(X)$ of the form given above to be a potential operator, it must satisfy the consistency condition [24]. The FI $J(X)$ for equation (1) can be written down using the formula given by Tonti [25,26], and choosing the boundary on X' to be such that the boundary terms vanish, we get the FI in the form

$$J(X) = \int \frac{1}{2}\left[-X'^2 + D_1X^2 + \frac{1}{2}D_3X^4\right] d\tau. \tag{3}$$

Thus the Lagrangian L, leading to equation (1), is given by:

$$L = \frac{1}{2}\left[-X'^2 + D_1X^2 + \frac{1}{2}D_3X^4\right]. \tag{4}$$

For which the Euler-Lagrange equation as

$$\frac{\partial L}{\partial X} = \frac{d}{d\tau}\left(\frac{\partial L}{\partial X'}\right), \tag{5}$$

Equation (5) yields us equation (1). In order to prove the invariance of the fundamental FI $\left(\int L d\tau\right)$, we look for a one-parameter infinitesimal group of transformations of the form:

$$\begin{aligned} \overline{\tau} &= \tau + \varepsilon\beta(\tau, X) + o\left(\varepsilon^2\right), \\ \overline{X} &= X + \varepsilon\gamma(\tau, X) + o\left(\varepsilon^2\right). \end{aligned} \tag{6}$$

The necessary condition for the fundamental FI $\left(\int L d\tau\right)$ to be invariant under the one-parameter infinitesimal group of transformation (6) is given by [27]:

$$\beta\frac{\partial L}{\partial \tau}+\gamma\frac{\partial L}{\partial X}+\frac{\partial L}{\partial X'}\left[\frac{\partial \gamma}{\partial \tau}+\left(\frac{\partial \gamma}{\partial X}-\frac{\partial \beta}{\partial \tau}\right)X'-\frac{\partial \beta}{\partial X}X'^2\right]+L\left[\frac{\partial \beta}{\partial \tau}+\frac{\partial \beta}{\partial X}X'\right]=0. \quad (7)$$

On substituting for L and its derivatives in equation (7) and collecting in descending order the coefficients of various powers of X' and setting these coefficients equal to zero, we obtain:

$$\frac{\partial \beta}{\partial X}=0; \qquad \frac{\partial \gamma}{\partial \tau}=0; \qquad \frac{\partial \gamma}{\partial X}-\frac{1}{2}\frac{\partial \beta}{\partial \tau}=0;$$
$$\left(D_1X+D_3X^3\right)\gamma+\frac{1}{4}\left(2D_1X^2+D_3X^4\right)\frac{\partial \beta}{\partial \tau}=0. \quad (8)$$

On solving the system of equations (8), we get the following expressions for β and γ:

$$\beta=c_1 \qquad \text{and} \qquad \gamma=0,$$

where c_1 is an arbitrary constant. Thus, the one-parameter infinitesimal group of transformation (6) takes the form:

$$\overline{\tau}=\tau+\varepsilon c_1+o\left(\varepsilon^2\right), \qquad \overline{X}=X+o\left(\varepsilon^2\right). \quad (9)$$

The solution for X of equation (1) is found by inverting it to an elliptic integral as:

$$\int_{X_0}^{X}\frac{dX}{\sqrt{C'+BX^2-X^4}}=\pm\sqrt{D_3/2}\tau, \quad (10)$$

where C', $B=-2D_1/D_3$ are constants. (For the numerical values $D_1=-1$, $D_3=3/2$, we have simply $B=2/3$.) Then the solution takes the form

$$X(\tau)=\sqrt{\alpha_1}\left[1-\left(1-\frac{\alpha_1}{\alpha_2}\right)sn^2\left(\sqrt{D_3/2}\tau,\varkappa\right)\right]^{\frac{1}{2}}. \quad (11)$$

where

$$\varkappa=\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_1}}, \qquad \alpha_{1,2}=\frac{B}{2}\pm\sqrt{\frac{B^2}{4}+C'}.$$

Furthermore, various choices of the values of the constant C, may lead to many subclasses of equation (10), e.g., if $C'\longrightarrow 0$, $\alpha_1\longrightarrow B$, $\alpha_2\longrightarrow 0$, $\varkappa\longrightarrow 1$. We get the solution of equation (1) in the latter case as

$$X=\sqrt{B}\ \text{sech}\left(\sqrt{D_3B/2}\tau\right). \quad (12)$$

Figure 1 shows an example of $U(\tau)$, where U is given later (Eq. (14)) with given values of the parameters in the interval $-1<\tau<1$. The solution (12) represents a bell shaped stable soliton for given values of the parameters as displayed in Figure 1.

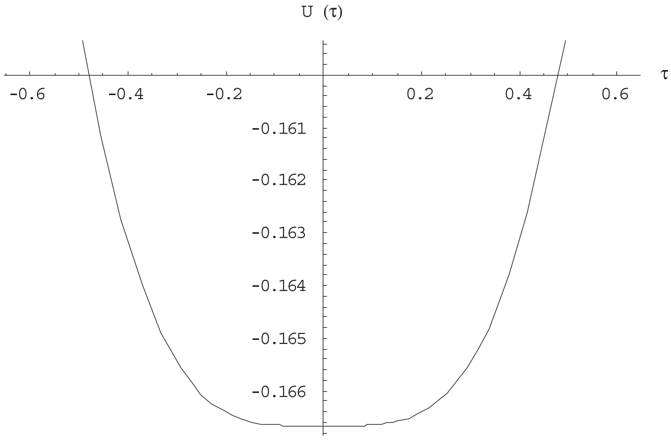

Fig. 1. The nonlinear potential energy $U(X)$ corresponding to the parameters $D_1=-1$, $D_3=1.5$, and $C=0.5$ in the interval $[-1,1]$ with equation (12).

3 Formulation of the variational principle

In the previous section, we saw that the linear problem is indifferent to the sign of the axial wave number k. It is obvious that, due to the symmetry of the configuration, this must hold also for the coefficient of the nonlinear term D_3 in equation (1). As a consequence, equation (1) describes the NLE of the stagnation line instability with either positive or negative helical polarization. However, the most general form of the linear eigenmode is a superposition of all eigenmodes with the same eigenvalue, i.e. in this case the eigenmodes with positive and negative helical polarization. From now on we therefore consider a perturbation of the form $X(t)\cos(\varphi+kz)+Y(t)\cos(\varphi-kz)$. Except for the possibility of mode rotation [9], this represents the most general form of the linear $m=1$ eigenmode in the plasma configuration under consideration. If the amplitudes X and Y in the expression above both are of order δ, it follows that the nonlinear interaction between X and Y produces a term proportional to XY^2 in equation (1). Similarly, in the corresponding equation for the time evolution of Y, a term involving YX^2 appears. Furthermore, the coefficients in front of XY^2 and YX^2 have to be equal, due to symmetry. This leads to the following CNLODEs describing the NLE of stagnation line instability with arbitrary polarization [12]

$$\frac{d^2X}{d\tau^2}+D_1X+D_3X^3+CXY^2=0,$$
$$\frac{d^2Y}{d\tau^2}+D_1Y+D_3Y^3+CYX^2=0. \quad (13)$$

As explained in relation to equation (2) we have no purely quadratic terms. Hence the mixing of X and Y yields cubic terms: a linear term in X affected by Y^2 in the equation for X and vice versa for the equation in Y. The coefficient C is, similarly to D_3, a quantity of order unity, which may be different for different marginal wave numbers of the instability, $C=C(k_c)$. The components $X(\tau)$ and $Y(\tau)$ of the Jost functions must be exactly equal to zero, respectively, to the left and to the right of the support, in order

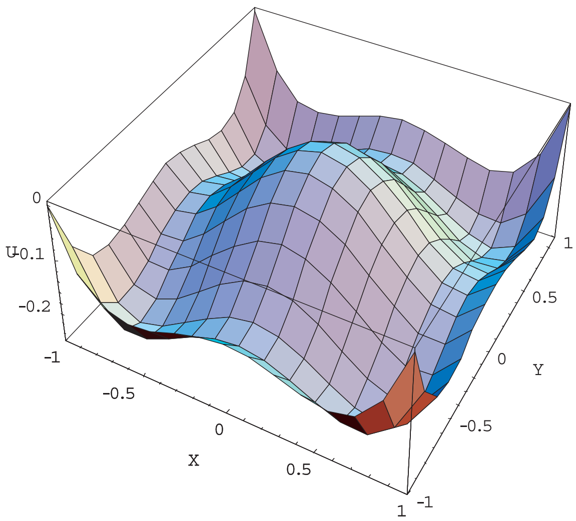

Fig. 2. (Color online) The nonlinear potential energy $U(X,Y)$ corresponding to the parameters $D_1 = -1$, $D_3 = 1.5$, and $C = 0.5$ in the interval $[-1,1]$ and $[-1,1]$ in equation (14).

to comply with the standard boundary condition for the Jost functions at infinity for bound states. This is another feature needs to be taken into account when choosing any trial functions, as well as the obvious condition that the Jost functions must be continuous. A similar technique can be used in order to illustrate the properties of the system (13). We notice in particular that the system has an energy $K+U = constant$, where K and U stand for the kinetic and the potential energy respectively, and given by

$$K = \frac{1}{2}\left(X\frac{d^2X}{d\tau^2} + Y\frac{d^2Y}{d\tau^2}\right);$$
$$U = \frac{1}{2}D_1\left(X^2+Y^2\right) + \frac{1}{4}D_3\left(X^4+Y^4\right) + \frac{1}{2}CX^2Y^2. \tag{14}$$

The evolution of the system is then described by a particle in a two-dimensional potential and the equation of motion $R_{\tau\tau} + \nabla U = 0$, with $R(X,Y)$. Figure 2 shows an example of $U(X,Y)$, with the parameters $D_1 = -1$, $D_3 = 1.5$, $C = 0.5$. Assuming that $D_1 < 0$, i.e., that we are on the unstable side of a marginal number k_c. Then, the equilibrium point $X = Y = U = 0$ corresponds to a maximum of $U(X,Y)$, as illustrated in Figure 2. Moreover, the first and the second variations of the obtained FI (i.e. direct problem of the CV) are also carried out and led to the same results of the stability criteria given in [12]. The condition for nonlinear stability of the system is that U increases in all directions in the XY-plane as $|X|, |Y| \to \infty$. This is the case if both D_3 and $C+D_3$ are positive. In this case, the equilibrium point at the origin, have eight equilibrium points in the XY-plane. Furthermore, these eight equilibria can be grouped into two classes. In the first class, either X or Y is zero, corresponding to helical polarization of the equilibrium, whereas in the second class $|X| = |Y|$, corresponding to plane polarization of the equilibrium. One example in the first class of equilibria is given by

$$X = \sqrt{-\frac{D_1}{D_3}}, \qquad Y = 0, \qquad U = -\frac{D_1^2}{4D_3}.$$

An example in the second class of equilibria is given by

$$X = Y = \sqrt{-\frac{D_1}{C+D_3}}, \qquad U = -\frac{D_1^2}{2C+2D_3}.$$

It is obvious that either the equilibria in the first class are stable and those in the second class unstable, or those in the second class are stable and the equilibria in the first class are unstable. For instance, the plane polarized equilibria in Figure 2 are stable, whereas the helically polarized equilibria are unstable [12].

We discuss the existence of a Lagrangian and the IVP for equation (13). In order to reduce equation (13) to a system of CNLODEs, we express it in the following form:

$$N(X,Y) = \frac{d^2X}{d\tau^2} + D_1X + D_3X^3 + CXY^2 = 0, \tag{15}$$

$$M(X,Y) = \frac{d^2Y}{d\tau^2} + D_1Y + D_3Y^3 + CYX^2 = 0, \tag{16}$$

where $X = X(\tau)$ and $Y = Y(\tau)$. The consistency conditions are expressed as follows [13–21, 25–28]

$$\begin{aligned}
\frac{\partial N}{\partial X_t} &= \frac{\partial}{\partial t}\left(\frac{\partial N}{\partial X_{tt}}\right) + \frac{1}{2}\frac{\partial}{\partial \tau}\left(\frac{\partial N}{\partial X_{\tau t}}\right),\\
\frac{\partial M}{\partial X} &= \frac{\partial N}{\partial Y} - \frac{\partial}{\partial \tau}\left(\frac{\partial N}{\partial Y_\tau}\right) - \frac{\partial}{\partial t}\left(\frac{\partial N}{\partial Y_t}\right) + \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial N}{\partial Y_{\tau\tau}}\right)\\
&\quad + \frac{\partial^2}{\partial \tau \partial t}\left(\frac{\partial N}{\partial Y_{t\tau}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial N}{\partial Y_{tt}}\right),\\
\frac{\partial M}{\partial X_\tau} &= -\frac{\partial N}{\partial Y_\tau} + 2\frac{\partial}{\partial \tau}\left(\frac{\partial N}{\partial Y_{\tau\tau}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial N}{\partial Y_{t\tau}}\right),\\
\frac{\partial M}{\partial X_t} &= -\frac{\partial N}{\partial Y_t} + 2\frac{\partial}{\partial t}\left(\frac{\partial N}{\partial Y_{tt}}\right) + \frac{\partial}{\partial \tau}\left(\frac{\partial N}{\partial Y_{\tau t}}\right),\\
\frac{\partial M}{\partial X_{\tau t}} &= \frac{\partial N}{\partial Y_{t\tau}}, \quad \frac{\partial M}{\partial X_{tt}} = \frac{\partial N}{\partial Y_{tt}},\\
\frac{\partial M}{\partial Y_\tau} &= \frac{\partial}{\partial \tau}\left(\frac{\partial M}{\partial Y_{\tau\tau}}\right) + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial M}{\partial Y_{\tau t}}\right),\\
\frac{\partial M}{\partial X_{\tau\tau}} &= \frac{\partial N}{\partial Y_{\tau\tau}}, \quad \frac{\partial M}{\partial Y_t} = \frac{\partial}{\partial t}\left(\frac{\partial M}{\partial Y_{tt}}\right) + \frac{1}{2}\frac{\partial}{\partial \tau}\left(\frac{\partial M}{\partial Y_{\tau t}}\right).
\end{aligned} \tag{17}$$

Furthermore, if the system of equation (13) satisfies the above conditions (17), then a FI $J(X,Y)$ can be written down using the formula given by Tonti [25,26], as

$$\begin{aligned}
J(X,Y) = \frac{1}{2}\int \Bigg[& X\frac{d^2X}{d\tau^2} + Y\frac{d^2Y}{d\tau^2} + D_1\left(X^2+Y^2\right)\\
& + \frac{1}{2}D_3\left(X^4+Y^4\right) + CX^2Y^2\Bigg] d\tau.
\end{aligned}$$

Thus, the Lagrangian L is given by

$$L(X,Y) = \frac{1}{2}\left[X\frac{d^2X}{d\tau^2} + Y\frac{d^2Y}{d\tau^2} + D_1\left[X^2+Y^2\right] + CX^2Y^2\right] + \frac{1}{4}D_3\left[X^4+Y^4\right]. \quad (18)$$

As a necessary check to our calculations we use the value of L in the Euler-Lagrange equations

$$\frac{\partial L}{\partial X} - \frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial X'}\right) + \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial L}{\partial X''}\right) = 0,$$
$$\frac{\partial L}{\partial Y} - \frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial Y'}\right) + \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial L}{\partial Y''}\right) = 0, \quad (19)$$

which yields us indeed the system of equations (13).

4 The rectangular box

We adopt the following ansatz for the X and Y functions, which is, as a matter of fact, the simplest possible choice:

$$X(\tau) = \begin{cases} 2\exp(-\mu(\tau-1)), & \tau > 1 \\ \tau+1, & |\tau| < 1 \\ 0, & \tau < -1 \end{cases} \quad (20)$$

$$Y(\tau) = \begin{cases} 0, & \tau > 1 \\ 1-\tau, & |\tau| < 1 \\ 2\exp(\mu(\tau+1)), & \tau < -1 \end{cases}. \quad (21)$$

In equations (20) and (21) we have one free parameter μ. In fact any choice of a trial function in this region will work, provided it vanishes at infinity. We remark that if we had allowed the values of X for $\tau < -1$ and of Y for $\tau > 1$ to be varied, then this would not be so. A straightforward calculation yields the values of the Lagrangian calculated with the trial functions (20) and (21) into the VP, we obtain the reduced variational problem as

$$J(X,Y) = \int_{-\infty}^{-1} L_G d\tau + \int_{-1}^{1} L_G d\tau + \int_{1}^{\infty} L_G d\tau. \quad (22)$$

Since

$$\int_{-\infty}^{-1} L_G d\tau = \int_{1}^{\infty} L_G d\tau = \frac{D_1}{\mu} + \frac{D_3}{\mu} + \mu,$$
$$\int_{-1}^{1} L_G d\tau = \frac{8}{3}D_1 + \frac{16}{5}D_3 + \frac{8}{15}C,$$

we have

$$J(X,Y) = 2\mu + 2\frac{D_1}{\mu} + 2\frac{D_3}{\mu} + \frac{8}{3}D_1 + \frac{16}{5}D_3 + \frac{8}{15}C. \quad (23)$$

Variation of the expression (23) in μ, leads to a quadratic equation

$$\mu^2 = (D_1 + D_3),$$

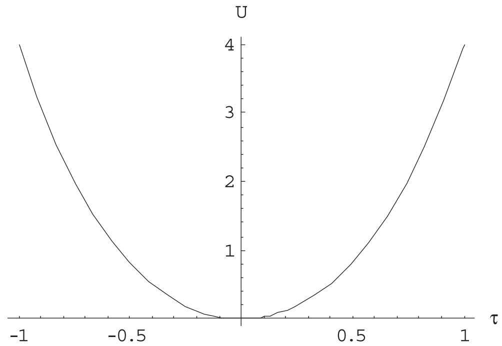

Fig. 3. The nonlinear potential energy $U(\tau)$ corresponding to the parameters $D_1 = -1, D_3 = 1.5$, and $C = 0.5$ in the interval $[-1, 1]$ in equation (14).

which has the roots

$$\mu = \sqrt{0.5}. \quad (24)$$

By insertion of this value of μ into the expression (23) yields the eventual result

$$J(X,Y) = 5.22843. \quad (25)$$

Figure 3 shows an example of $U(\tau)$ with the parameters $D_1 = -1, D_3 = 1.5$, and $C = 0.5$ in the interval $|\tau| < 1$.

5 The two box potential

For the next example of the application of the variational approximation, it is natural to try a generalization of the linear ansatz of equations (20) and (21), which allows discontinuity of the first derivatives of the Jost functions as:

$$X(\tau) = \begin{cases} (1+\alpha)\exp(-\mu\ (\tau-1)), & \text{at} \quad \tau > 1 \\ 1+\alpha\tau, & \text{at} \quad 0 < \tau < 1 \\ \tau+1, & \text{at} \quad -1 < \tau < 0 \\ 0, & \text{at} \quad \tau < -1 \end{cases} \quad (26)$$

$$Y(\tau) = \begin{cases} 0, & \text{at} \quad \tau > 1 \\ 1-\tau, & \text{at} \quad 0 < \tau < 1 \\ 1-\alpha\tau\ , & \text{at} \quad -1 < \tau < 0 \\ (1+\alpha)\exp(\mu(\tau+1)), & \text{at} \quad \tau < -1 \end{cases}. \quad (27)$$

Here α is a variational parameter. For $|\tau| > 1$, the Jost function are assumed to have the same form as in equations (20) and (21), but, as mentioned before, the Lagrangian given by equation (22) is insensitive to the structure of the Jost functions outside the box. This ansatz now contains two nontrivial variational parameters μ and α. Substituting from equations (26) and (27) into equation (22), one can find the values of the integrals $J(X,Y)$ as:

$$J(X,Y) = \int_{-\infty}^{-1} L_G d\tau + \int_{-1}^{0} L_G d\tau + \int_{0}^{1} L_G d\tau + \int_{1}^{\infty} L_G d\tau.$$

Since

$$\int_{-1}^{0} L_G d\tau = \int_{0}^{1} L_G d\tau = \frac{2}{3}D_1 + \frac{3}{10}D_3 + \frac{1}{6}C + \left(\frac{1}{2}D_1 + \frac{1}{2}D_3 + \frac{1}{12}C\right)\alpha + \left(\frac{1}{6}D_1 + \frac{1}{2}D_3 + \frac{1}{60}C\right)\alpha^2 + \frac{1}{4}D_3\alpha^3 + \frac{1}{20}D_3\alpha^4,$$

$$\int_{-\infty}^{-1} L_G d\tau = \int_{1}^{\infty} L_G d\tau = \frac{(1+\alpha)^2(D_3(1+\alpha)^2 + 4(D_1+\mu^2))}{16\mu}$$

we obtain

$$J(X,Y) = \frac{4}{3}D_1 + \frac{3}{5}D_3 + \frac{1}{3}C + \left(D_1 + D_3 + \frac{1}{6}C\right)\alpha + \left(\frac{1}{3}D_1 + D_3 + \frac{1}{30}C\right)\alpha^2 + \frac{1}{2}D_3\alpha^3 + \frac{1}{10}D_3\alpha^4 + \frac{(1+\alpha)^2(D_3(1+\alpha)^2 + 4(D_1+\mu^2))}{8\mu}.$$

with $D_1 = -1$, $D_3 = 3/2$ and $C = 1/2$, we have

$$J(X,Y) = -0.27 + 0.58\alpha + 1.18\alpha^2 + 0.75\alpha^3 + 0.15\alpha^4 + \frac{(1+\alpha)^2(-2.5 + 3\alpha + 1.5\alpha^2 + 4\mu^2)}{8\mu}. \quad (28)$$

Varying the expression (28) with respect to μ and α, we get

$$(1+\alpha)^2 - \frac{(1+\alpha)^2(-2.5 + 3\alpha + 1.5\alpha^2 + 4\mu^2)}{8\mu^2} = 0,$$

$$0.058 + 2.37\alpha + 2.25\alpha^2 + 0.6\alpha^3 + \frac{(1+\alpha)^2(3+3\alpha)}{8\mu} + \frac{(1+\alpha)(-2.5 + 3\alpha + 1.5\alpha^2 + 4\mu^2)}{4\mu} = 0. \quad (29)$$

The roots of this equation are

$$\alpha = -1.01129 - 0.065585i, \quad \mu = 0.000277381 + 1.00078i. \quad (30)$$

By substitution of the roots of α and μ into the expression for the Lagrangian J produces analytical expressions like

$$J = -0.265925 + 0.00435777i. \quad (31)$$

Figures 4a and 4b show an example of $U(\tau)$, with the parameters $D_1 = -1, D_3 = 1.5$, and $C = 0.5$ in the interval $0 < \tau < 1$ and $-1 < \tau < 0$, respectively. Here we note that the latter figures represents stable system.

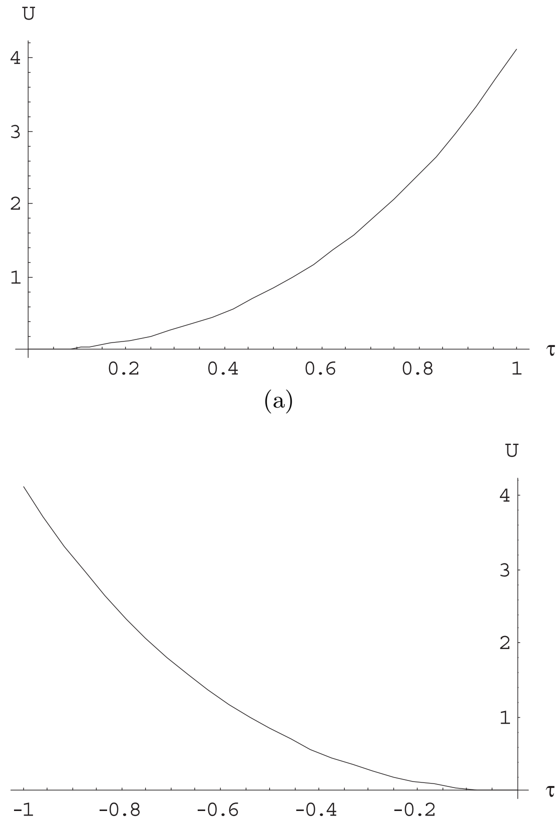

Fig. 4. The nonlinear potential energy $U(\tau)$ corresponding to the parameters $D_1 = -1$, $D_3 = 1.5$, and $C = 0.5$ in the interval $[0,1]$ with $|\alpha| = 1.01341$ for (a) and in the interval $[-1,0]$ with $|\alpha| = 1.01341$ for (b), in equation (14).

6 General case

Qualitatively similar results for the two box potential can be obtained on the basis of a different ansatz where we approximate the Jost functions by polynomials of order n instead of the piecewise linear function in equations (20) and (21) for $|\tau| < 1$

$$X(\tau) = \begin{cases} \sum_{m=1}^{N} C_m 2^m \exp(-\mu\ (\tau-1)), & \text{at} \quad \tau > 1 \\ \sum_{m=1}^{N} C_m (1+\tau)^m, & \text{at} \quad |\tau| < 1 \\ 0, & \text{at} \quad \tau < -1 \end{cases} \quad (32)$$

$$Y(\tau) = \begin{cases} 0, & \text{at} \quad \tau > 1 \\ \sum_{m=1}^{N} C_m (1-\tau)^m, & \text{at} \quad |\tau| < 1 \\ \sum_{m=1}^{N} C_m 2^m \exp(\mu\ (\tau+1)) & \text{at} \quad \tau < -1 \end{cases} \quad (33)$$

where $C_1 = 1$ and C_N are the variational parameters. Substituting equations (32) and (33) into equation (22), one can find the values of the integrals $J(X,Y)$

$$J(X,Y) = \int_{-\infty}^{-1} L_G d\tau + \int_{-1}^{1} L_G d\tau + \int_{1}^{\infty} L_G d\tau,$$

$$J(X,Y)=\mu\sum_{m=1}^{N}m(m-1)C_m\sum_{n=1}^{N}C_n2^{m+n-1}+\frac{2D_1}{\mu}\left(\sum_{m=1}^{N}C_m2^{m-1}\right)^4+\sum_{m=1}^{N}m(m-1)C_m\sum_{n=1}^{N}C_n\left[\frac{2^{m+n-1}}{m+n-1}\right]$$
$$+D_1\sum_{m=1}^{N}C_m\sum_{n=1}^{N}C_n\left[\frac{2^{m+n+1}}{m+n+1}\right]+D_3\sum_{m=1}^{N}C_m\sum_{n=1}^{N}C_n\sum_{i=1}^{N}C_i\sum_{j=1}^{N}C_j\left[\frac{2^{m+n+i+j}}{m+n+i+j+1}\right]+\frac{2D_3}{\mu}\left(\sum_{m=1}^{N}C_m2^{m-1}\right)^2$$
$$+\left[\frac{C}{2}\sum_{m=1}^{N}C_m\sum_{j=1}^{N}C_j\sum_{n=1}^{N}C_n\sum_{i=1}^{N}C_i2^{m+j+n+i}\right]\left[\frac{\Gamma(j+i+1)\Gamma(m+n+1)}{\Gamma(n+j+m+i+2)}\right]. \quad (34)$$

then we have

see equation (34) above.

Varying the expression (34) with respect to $C_m\,(m=2,3,...,N)$, yields N equations in N unknowns. By solving these equations we get the value of $C_m\,(m=2,3,...,N)$. Lastly, insertion of these values of $C_m\,(m=2,3,...,N)$ into the expression (34) we get the approximation of the Lagrangian $L(X,Y)$.

6.1 Example

We choose the trial function in the form

$$X(\tau)=\begin{cases}(2+4C_2)\exp(-\mu\,(\tau-1)), & \text{at} \quad \tau>1\\ 1+\tau+C_2(1+\tau)^2, & \text{at} \quad |\tau|<1\\ 0, & \text{at} \quad \tau<-1\end{cases} \quad (35)$$

$$Y(\tau)=\begin{cases}0, & \text{at} \quad \tau>1\\ 1-\tau+C_2(1-\tau)^2, & \text{at} \quad |\tau|<1\\ (2+4C_2)\exp(\mu(\tau+1)) & \text{at} \quad \tau<-1\end{cases}. \quad (36)$$

Then we have

$$J(X,Y)=\frac{2(1+2C_2)^2(D_1+D_3(1+2C_2)^2+\mu^2)}{\mu}+D_1\left(\frac{8}{3}+8C_2+\frac{32}{5}C_2^2\right)+D_3\left(\frac{16}{5}+\frac{64}{3}C_2+\frac{384}{7}C_2^2+64C_2^3+\frac{256}{9}C_2^4\right)+C\left(\frac{8}{15}+\frac{32}{15}C_2+\frac{64}{21}C_2^2+\frac{64}{35}C_2^3+\frac{128}{315}C_2^4\right). \quad (37)$$

Varying the expression (37) with respect to μ and C_2, we get

$$8\left(0.5+C_2^2\right)-\frac{48(0.092+C_2)(0.91+C_2)(0.25+C_2(1+C_2))}{\mu^2}=0;$$

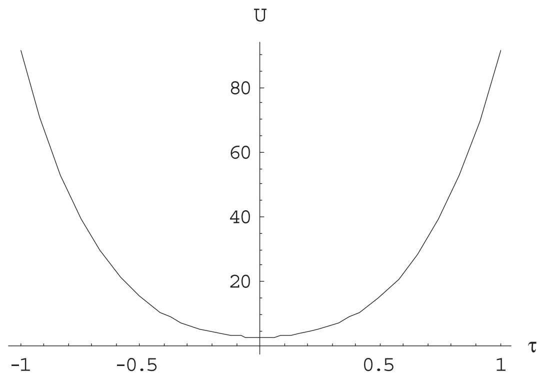

Fig. 5. The nonlinear potential energy $U(\tau)$ corresponding to the parameters $D_1=-1, D_3=1.5$, and $C=0.5$ in the interval $[-1,1]$ with $|C_2|=0.509186$, in equation (14).

$$\frac{1}{\mu}(8\,(0.89+\mu)\,(2.24+\mu)+16C_2\,(0.91+\mu)\,(8.76+\mu)+4C_2^2\,(48+42.87\mu)+3C_2^3\,(96+96.91\mu)=0. \quad (38)$$

The roots of this equation are

$$\mu=0.0016+1.0031i, \qquad C_2=-0.5081-0.033i. \quad (39)$$

By substitution of the roots of C_2 into the expression for the Lagrangian L, yields the eventual results as, respectively (the imaginary part is the same as in Eq. (25))

$$J=-0.2116+0.01795i. \quad (40)$$

Figure 5 shows an example of $U(\tau)$, with the parameters $D_1=-1, D_3=1.5$, and $C=0.5$ in the interval $-1<\tau<1$ with $|C_2|=0.509186$. A similar argument as before applies quite well for figures which corresponds to stable state.

7 Possible application to solar filaments

The above theory was developed in a general way, aiming mainly at laboratory plasmas. However, the possible application to the various kinds of solar filaments may be discussed briefly. Filament bands, consisting of enchained filaments and filament channels encircle the whole Sun, and situated some 10 000 to 30 000 km above the solar surface. They are of particular relevance for the solar cycle [29,30]. They correspond to a neutral field line. However, neutral means here essentially: in the line of sight. In fact the filaments may still have a small magnetic field (a few gauss) along their central line. Probably this small field acts as a backbone and contributes to a fair stability, allowing eruptions from time to time. It is felt that the filament band as a whole corresponds somewhat to the stagnation line considered in the present paper. However, the filament band may evolve such that it has nearly a vanishing field at certain places, where the instability may set in and cause eruptions, as for the stagnation line under study. The conservation of flux would then require some deviation of the central field line from the plasma tube, like a plunging in the Sun or get lost due to resistivity.

There occur circular filaments also, other than the huge filament bands, encircling a smaller domain. Again they are neutral along the line of sight and again there is a small field along them: this may be smaller than in the case of the filament bands and in fact the stability is less good: eruptions and destruction are more frequent. Still the approximation is expected to be modest in general.

Finally there occur more or less straight filaments of various shapes and kinds. Their stability is still poorer than the previous ones, so that their backbone field may be expected to be still weaker. Often they are studied as a current sheet or a current line. However, some of them may be described by a neutral sheet or a stagnation line, more or less as a limiting case. Those are in general weaker and are sooner destroyed by instability. It is our opinion that the theory developed above may at least be useful for this small rather limiting group of filaments.

Recently Kikuchi [31] has considered a quadripolar cusp geometry in relation with certain clouds and thunderstorms. As a cusp geometry may be reproduced in the laboratory this may reveal a good possibility to test the theory of the elliptic stagnation line in nature and in the laboratory.

8 Conclusion

Wahlberg and co-authors, in a series of papers [8–12], have shown that, near marginal stability, the NLE of the instability can be described in terms of a two-dimensional potential $U(X,Y)$, where X and Y represent the amplitudes of the perturbations with positive and negative helical polarization. They obtained the NLE of the stagnation line instability with either positive or negative helical polarization, they constructed the energy $K + U = constant$, where K the kinetic energy and U the potential energy for the specific physics at hand and they discussed the stability from the energy integral.

We started on the other hand from the EE obtained in [12] and the existence of a Lagrangian and an IVP in the sense of the inverse problem of CVs through deriving the FI corresponding to a given CNLODEs. The obtained result turned out to be in agreement with the one as given in [13–21]. Moreover, the first and the second variations of the obtained FI (i.e. direct problem of the CVs) are also carried out and led to be an extension to the results of the stability criteria as given in [12] to include bistable systems. The latter systems may prove to be of great interest particularly in nonlinear stability including solitons occurring in various applications (e.g. Rayleigh Taylor, Kelvin-Helmholtz instabilities and nonlinear evolution equations).

The existence and formulation of the VPs for the NLEE are given. The invariance identities involving the Lagrangian L and the generators of the infinitesimal Lie group of transformations have been utilized for writing down their first integrals via Noether's theorem [23,24]. We formulated the VP for the two CNLODEs describing the NLE of stagnation line instability and we formulated the Lagrangian L. We demonstrated the simplest example of the application of this technique, taking the box shaped initial pulse and an ansatz based on linear Jost functions in the region of localization of the box. We considered in more detail the limiting case of the above mentioned box with the phase jump equal to π, i.e., a combination of two boxes of opposite signs, the total area of the initial pulse being thus zero. We developed a variational approximation for finding the eigenvalues of this pulse, the Jost functions being approximated by a piecewise linear ansatz, which has two variational parameters. Next we approximated the Jost functions by polynomials of order n instead of a piecewise linear function.

We suggested that the theory may be useful for the study of some solar filaments, in particular the ones that are more or less straight and moreover fairly weak. A good candidate to test the theory may be given by a quadripolar cusp geometry in some clouds and thunderstorms as well as in the laboratory.

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